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On the laminar flow in a free jet of liquid at high Reynolds numbers

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This paper is concerned with the jet of liquid, open to the atmosphere, that emerges from a two-dimensional channel in which there is Poiseuille flow far upstream, the flow being driven by an applied pressure gradient. The problem is discussed with the aid of the method of matched asymptotic expansions; the small parameter involved is the inverse Reynolds number. A boundary layer forms adjacent to the free surface, and a classical boundary-layer analysis is applied to find the flow there (for moderate distances downstream); the influence of this boundary layer on the flow in the core of the jet is then investigated. Higher-order boundary-layer effects, such as indeterminacy and eigensolutions, are also discussed. The first few terms are found of an asymptotic expansion for the equation of the free surface, and considerations of momentum balance are applied to find the asymptotic contraction ratio of the jet.

1. Introduction

Consider the two-dimensional flow of an incompressible liquid along a channel, which we take to be semi-infinite (figure 1). The Reynolds number (based on the channel width) is large. We assume that the flow has the basic Poiseuille profile to lowest order, and consider how this is modified when the fluid leaves the end of the channel in the form of a jet. More interesting physically is the case of an axisymmetric jet leaving a circular pipe; this can be studied by methods similar to those described here, and it is hoped to publish the details elsewhere.

When the fluid detaches itself from the wall of the channel, the removal of the wall stress causes a boundary layer to form at the free surface; in this layer the parabolic velocity profile adjusts itself so as to satisfy the condition of zero stress at the free surface (surface tension is ignored). In the case of an inviscid liquid, this condition would not be imposed, and all the conditions of the problem would be satisfied by postulating that the parabolic profile continues unchanged in the jet region. However, no uniqueness theorem exists for this inviscid problem, and it is conceivable that other solutions might exist; nevertheless, it is assumed in this paper that Poiseuille flow everywhere is the proper inviscid limit. With this assumption, the flow in the interior of the jet is unaffected to lowest order, though the boundary layer can be expected to induce perturbations to it, and also to the flow upstream in the channel.

We attack the problem in a systematic manner by using the now standard method of matched asymptotic expansions, described by Van Dyke (1964).

The solution is developed in powers of ϵ, where ϵ^3 is an inverse Reynolds number, both in the 'inner' (boundary-layer) region and in the 'outer' region of the core; the two expansions are matched by standard procedures.

Goren (1966) considered the development of such a boundary layer when the basic flow is a simple shear flow and the flow field of infinite extent across the stream. He found that the boundary-layer thickness and the displacement of the free surface grow as the cube root of the downstream distance. We confirm Goren's solution as a first approximation for our boundary layer (to within the 2 % accuracy of his computations), find higher-order terms, and consider the perturbed flow in the core.

The main difficulty is that the boundary layer may interact with the main flow, in the sense that it may induce a pressure gradient there, and this effect depends crucially on the shape of the free surface, which is unknown at the outset. Goren assumed that there was no such interaction, so that in his solution modifications to the basic profile are confined to the boundary layer; the shape of the free surface is then found from the condition that the gain of volumetric flow rate in the boundary layer due to the speeding-up of the fluid there is balanced only by the loss due to the contraction of the jet, and not by any modification of the basic profile away from the boundary layers. We do not make this assumption, but find that, for matching to be possible, the first- and second-order perturbations to the main flow must vanish identically; and this justifies Goren's assumption, to second order. We can then deduce the equation of the free surface, again from the matching.

It should be made clear that the boundary-layer solution derived here will break down at large distances downstream, actually when $x^{-1} = O(\epsilon^3)$, when the boundary-layer thickness is no longer small compared to the width of the channel. This breakdown is exemplified by the fact that in our solution the jet does not tend to any asymptotic width. Harmon (1955) obtained for a circular jet the asymptotic contraction ratio $\sqrt{3}/2$ by considering the balance of mass transport and momentum along the jet, taking plug flow at infinity and a parabolic profile at the exit. This latter assumption is open to criticism; our analysis shows that it is valid at high Reynolds number, since the core flow is unaffected to first and second order, but otherwise it is inadequate; this conclusion is borne out by the experiments of Middleman & Gavis (1961). Using our solution to calculate the momentum flux in the jet we can, in principle, find a series expansion for the contraction ratio at high Reynolds number, the zero-order term being Harmon's value; we calculate the next term, and find that it vanishes.

The flow far downstream in the jet is discussed by Goren & Wronski (1966), and references to other related work can be found there and in Goren (1966).

2. The governing equations and boundary conditions

We take the $\tilde{x}$-axis along the lower edge of the channel, and the $\tilde{z}$-axis across the channel mouth (figure 1); the tilde indicates a dimensional quantity. If a is the width of the channel, the stream function of the basic Poiseuille flow is

$$\tilde{\psi}_0 = A(\tilde{z}^2 - \tfrac{2}{3}\tilde{z}^3/a), \tag{2.1}$$

where A is a constant.

We define an inverse Reynolds number $\epsilon_* = \nu/Aa^2$, where ν is the kinematic viscosity. This will be our small parameter in what follows.

For convenience later a variable $\tilde{y}$ is defined by

$$\tilde{y} = \tilde{z} - \tilde{\zeta}(\tilde{x}, \epsilon_*), \tag{2.2}$$

where $\tilde{z} = \tilde{\zeta}(\tilde{x}, \epsilon_*)$ is the equation of the lower free streamline.

Before setting up the equations we introduce non-dimensional quantities by measuring lengths with respect to a and velocities with respect to Aa; that is, we write $\tilde{x} = ax$, etc., and $\tilde{\psi} = Aa^2\psi$. (2.1) now becomes

$$\psi_0 = z^2 - \tfrac{2}{3}z^3, \tag{2.3}$$

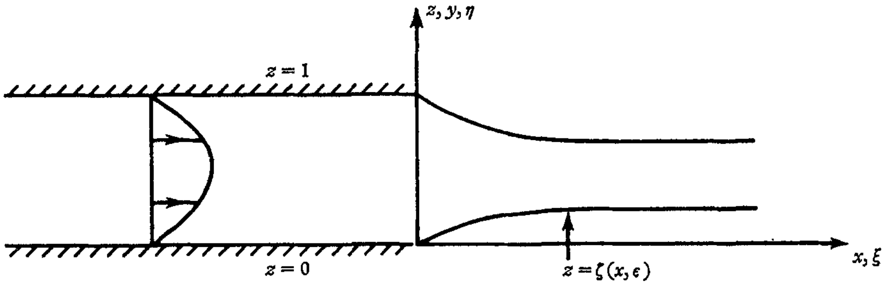

FIGURE 1. Notation.

and the Navier–Stokes equations for steady laminar flow are, in the new variables,

$$\psi_z\psi_{xz} - \psi_x\psi_{zz} = -P_x + \epsilon_*(\psi_{xxz} + \psi_{zzz}), \tag{2.4}$$

$$-\psi_z\psi_{xx} + \psi_x\psi_{xz} = -P_z - \epsilon_*(\psi_{xxx} + \psi_{xzz}). \tag{2.5}$$

Here P is the non-dimensional pressure (real pressure divided by $\rho A^2 a^2$, ρ being the density).

For $x > 0$ the boundary conditions to be applied on the lower free surface $z = \zeta(x)$† are

$$\psi = 0, \tag{2.6}$$

$$P_{ij}n_j = 0, \tag{2.7}$$

where P_{ij} are the components of the non-dimensional stress tensor and $\mathbf{n}$ is the normal to the surface. With

$$\mathbf{n} = (-\zeta'(x), 1)/[\zeta'^2(x) + 1]^{\frac{1}{2}}, \tag{2.8}$$

$$P_{ij} = -P\delta_{ij} + \epsilon_*\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.9}$$

(2.7) becomes

$$P + \epsilon_*\{2\psi_{xz} + \zeta'(x)(\psi_{zz} - \psi_{xx})\} = 0, \tag{2.10}$$

$$\zeta'(x)P + \epsilon_*\{\psi_{zz} - \psi_{xx} - 2\zeta'(x)\psi_{xz}\} = 0. \tag{2.11}$$

The other conditions to be satisfied are

$$\psi = \tfrac{1}{6} \quad \text{on} \quad z = \tfrac{1}{2}, \tag{2.12}$$

$$\psi_z = 0 \quad \text{on} \quad z = 0, 1, \tag{2.13}$$

$$\psi \to z^2 - \tfrac{2}{3}z^3 \quad \text{as} \quad x \to -\infty. \tag{2.14}$$

We consider z in the range $0 \leqslant z \leqslant \frac{1}{2}$; the flow for $\frac{1}{2} \leqslant z \leqslant 1$ is obtained from symmetry considerations.

† The dependence of ζ on ϵ_* will not be displayed when it is not relevant to the point under discussion.

3. The inner expansion

To examine the boundary-layer structure we change the scaling in the transverse direction by writing $y = \epsilon\eta$, where $\epsilon = \epsilon_*^{\alpha}$ $(\alpha > 0)$ and α is to be determined. Anticipating that the displacement ζ of the free surface is of the same order of magnitude as the boundary-layer thickness, we write $\zeta(x,\epsilon) = \epsilon h(x,\epsilon)$ and henceforth work with h. It is not necessary to assume that $h(x,\epsilon) = O(1)$ as $\epsilon \to 0$; examination of (3.2) below shows that the inner expansion developed in this section holds provided only that $h = o(\epsilon^{-1})$, i.e. ζ tends to 0 with ϵ. The work of § 5 then shows that we do indeed have $h = O(1)$.

Making the change of independent variables

$$\left.\begin{aligned} x &= \xi, \\ z &= \epsilon(\eta + h(\xi)) \end{aligned}\right\} \tag{3.1}$$

in (2.4) and (2.5), we obtain

$$\begin{aligned}\psi_\eta \psi_{\xi\eta} - \psi_\xi \psi_{\eta\eta} = {}& -\epsilon^2(P_\xi - h'P_\eta) + \epsilon^{1/\alpha - 1}\psi_{\eta\eta\eta} \\ & + \epsilon^{1/\alpha+1}(\psi_{\xi\xi\eta} - h''\psi_{\eta\eta} - 2h'\psi_{\xi\eta\eta} + h'^2\psi_{\eta\eta\eta}),\end{aligned} \tag{3.2}$$

$$\begin{aligned}&-\psi_\eta\psi_{\xi\xi} + \psi_\xi\psi_{\xi\eta} + h''\psi_\eta^2 + h'(\psi_\eta\psi_{\xi\eta} - \psi_\xi\psi_{\eta\eta}) \\ &\quad = -P_\eta - \epsilon^{1/\alpha+1}\left(\frac{\partial}{\partial\xi} - h'\frac{\partial}{\partial\eta}\right)^3\psi - \epsilon^{1/\alpha-1}(\psi_{\xi\eta\eta} - h'\psi_{\eta\eta\eta}).\end{aligned} \tag{3.3}$$

Note that x and ξ are distinguished only in differentiation.

We seek a solution of these equations in the form of an 'inner expansion' in ϵ. In order to match this to the outer Poiseuille flow, we must have $\psi \sim y^2$ as $\eta \to \infty$ in the inner region, to lowest order in ϵ, so ψ must be of order ϵ^2. In order to achieve a balance between the viscous terms and the inertial terms in (3.2) we must therefore take $\alpha = \frac{1}{3}$. That is, $\epsilon = (\nu/Aa^2)^{\frac{1}{3}}$.

Our inner expansion for ψ begins with a term in ϵ^2; we assume until we have evidence to the contrary that it proceeds in powers of ϵ, so that

$$\psi = \epsilon^2\Psi_2(\xi,\eta) + \epsilon^3\Psi_3(\xi,\eta) + \dots. \tag{3.4}$$

Similarly we expand h and P as

$$h = \epsilon^{-1}\zeta = h_0(x) + \epsilon h_1(x) + \dots, \tag{3.5}$$

$$P = P_0(\xi,\eta) + \epsilon P_1(\xi,\eta) + \dots. \tag{3.6}$$

The boundary conditions on the free surface $\eta = 0$ are, from (2.6), (2.10) and (2.11),

$$\psi = 0, \tag{3.7}$$

$$P + \epsilon^2(2\psi_{\xi\eta} - h'\psi_{\eta\eta}) - \epsilon^4 h'\left(\frac{\partial}{\partial\xi} - h'\frac{\partial}{\partial\eta}\right)^2\psi = 0, \tag{3.8}$$

$$\epsilon(h'P + \psi_{\eta\eta}) - \epsilon^3\left\{\left(\frac{\partial}{\partial\xi} - h'\frac{\partial}{\partial\eta}\right)^2\psi + 2h'(\psi_{\xi\eta} - h'\psi_{\eta\eta})\right\} = 0. \tag{3.9}$$

From (3.3) and (3.8) we conclude at once that P is of order ϵ^4. To the lowest order in ϵ, (3.2) then becomes

$$\Psi_{2\eta}\Psi_{2\xi\eta} - \Psi_{2\xi}\Psi_{2\eta\eta} = \Psi_{2\eta\eta\eta}, \tag{3.10}$$

and (3.7) and (3.9) give

$$\Psi_2(\xi,0) = \Psi_{2\eta\eta}(\xi,0) = 0. \tag{3.11}$$

To complete the system of equations for $\Psi_2(\xi,\eta)$ we require one further boundary condition. This is the matching condition

$$\Psi_2(\xi,\eta) \sim \eta^2 \quad \text{as} \quad \eta\to\infty, \tag{3.12}$$

which is derived formally in § 5.

Reverting temporarily to the dimensional variables $\tilde{x} = ax$, $\tilde{\psi} = Aa^2\psi$, etc., we have, to order ϵ^2,

$$\begin{aligned}\tilde{\psi} &= Aa^2\epsilon^2\Psi_2(\xi,\eta)\\ &= Aa^2(\nu/Aa^2)^{\frac{2}{3}}\,\Psi_2(\tilde{x}/a, A^{\frac{1}{3}}\tilde{y}/\nu^{\frac{1}{3}}a^{\frac{1}{3}}).\end{aligned} \tag{3.13}$$

This must be matched for $\eta\to\infty$ to the term $A\tilde{z}^2 \sim A\tilde{y}^2$ of the outer solution; thus with this approximation the right-hand side of (3.13) is independent of a, and we conclude that

$$\Psi_2(\tilde{x}/a, A^{\frac{1}{3}}\tilde{y}/\nu^{\frac{1}{3}}a^{\frac{1}{3}}) = (\tilde{x}/a)^{\frac{2}{3}}f_2((A^{\frac{1}{3}}\tilde{y}/\nu^{\frac{1}{3}}a^{\frac{1}{3}})/(\tilde{x}/a)^{\frac{1}{3}}),$$

i.e.

$$\Psi_2(\xi,\eta) = \xi^{\frac{2}{3}}f_2(\theta), \tag{3.14}$$

where $\theta = \eta/\xi^{\frac{1}{3}}$. θ is the similarity variable used by Goren (1966) in his solution, and by Goldstein (1930) in a different context.

The equation for $f_2(\theta)$ is, from (3.10),

$$f_2''' + \tfrac{2}{3}f_2f_2'' - \tfrac{1}{3}f_2'^2 = 0, \tag{3.15}$$

and the boundary conditions are

$$f_2(0) = f_2''(0) = 0, \tag{3.16}$$

$$f_2(\theta) \sim \theta^2 \quad \text{as} \quad \theta\to\infty. \tag{3.17}$$

An equation essentially the same as (3.15) was investigated by Goldstein (1930). For large θ, the solution has asymptotically the form

$$f_2(\theta) \sim A_2(\theta+c)^2 + O[\exp(-\tfrac{2}{9}A_2\theta^3)], \tag{3.18}$$

where A_2 and c are constants; this is derived in appendix A. We choose $A_2 = 1$ to satisfy (3.17); the constant c and a third arbitrary constant in the exponential term give us enough freedom to satisfy (3.16).

The results of numerical computation of $f_2(\theta)$ are given in table 1. The value of c is found to be 0·70798.

The surface speed to this order is given by

$$\begin{aligned}u(x,y=0) &= \epsilon x^{\frac{1}{3}}f_2'(0)Aa\\ &\simeq 2{\cdot}5572(A^2\nu\tilde{x})^{\frac{1}{3}},\end{aligned} \tag{3.19}$$

which agrees with the result obtained by Goren (1966, equation (28)) to within the 2 % accuracy of his calculations.

To the next order in ϵ, (3.2) becomes

$$\Psi_{2\eta}\Psi_{3\xi\eta} + \Psi_{3\eta}\Psi_{2\xi\eta} - \Psi_{2\xi}\Psi_{3\eta\eta} - \Psi_{3\xi}\Psi_{2\eta\eta} = \Psi_{3\eta\eta\eta}, \tag{3.20}$$

and the boundary conditions (3.7) and (3.9) become

$$\Psi_3(\xi, 0) = \Psi_{3\eta\eta}(\xi, 0) = 0. \tag{3.21}$$

The matching condition from §5,

$$\Psi_3(\xi, \eta) \sim -\tfrac{2}{3}\eta^3 \quad \text{as} \quad \eta \to \infty, \tag{3.22}$$

completes the specification of $\Psi_3(\xi, \eta)$.

In terms of the dimensional variables, this second term of the inner expansion matches to $-\frac{2}{3}A\tilde{z}^3/a$; thus the dependence on a comes in the form of a factor $1/a$, and a dimensional analysis analogous to that leading to (3.14) gives

$$\Psi_3(\xi, \eta) = \xi f_3(\theta). \tag{3.23}$$

θ	$f_2(\theta)$	$f_3(\theta)$	θ	$f_2(\theta)$	$f_3(\theta)$
0·2	0·5143	−0·7298	2·2	8·4563	−20·4700
0·4	1·0458	−1·5317	2·4	9·6595	−24·5078
0·6	1·6104	−2·4734	2·6	10·9427	−29·0433
0·8	2·2222	−3·6142	2·8	12·3059	−34·1082
1·0	2·8924	−5·0037	3·0	13·7491	−39·7343
1·2	3·6295	−6·6821	3·2	15·2723	−45·9538
1·4	4·4392	−8·6823	3·4	16·8755	−52·7985
1·6	5·3252	−11·0333	3·6	18·5587	−60·3005
1·8	6·2894	−13·7627	3·8	20·3219	−68·4918
2·0	7·3330	−16·8984	4·0	22·1651	−77·4044

TABLE 1. The functions $f_2(\theta)$ and $f_3(\theta)$

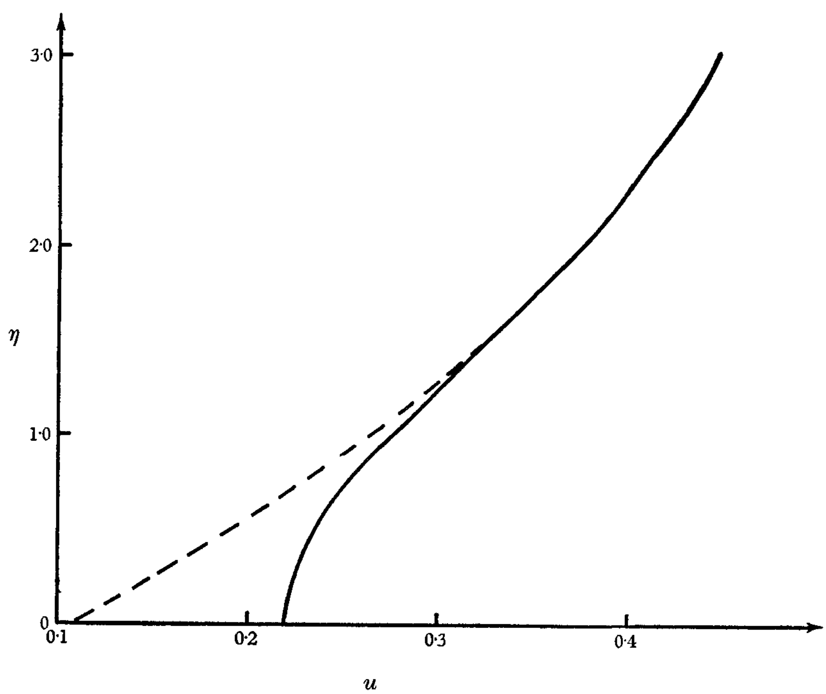

FIGURE 2. ——, the boundary-layer profile at $x = 1$ with $\epsilon = 0·1$ (Reynolds number 1000); – – – –, its asymptotic behaviour.

The equation and boundary conditions for $f_3(\theta)$ are

$$\left.\begin{aligned} f_3''' + \tfrac{2}{3}f_2f_3'' - f_2'f_3' + f_2''f_3 &= 0, \\ f_3(0) = f_3''(0) &= 0, \\ f_3(\theta) \sim -\tfrac{2}{3}\theta^3 \quad \text{as} \quad \theta &\to \infty. \end{aligned}\right\} \tag{3.24}$$

The asymptotic solution of this equation has the form

$$f_3 \sim A_3(t^3-3) + B_3 t + O[\exp(-\tfrac{2}{9}t^3)]; \tag{3.25}$$

here $t = \theta + c$ and A_3, B_3 are arbitrary constants. We choose $A_3 = -\frac{2}{3}$ and B_3 and a third arbitrary constant in the exponential term are at our disposal to fit the boundary conditions on $\eta = 0$. The function $f_3(\theta)$ is tabulated in table 1; the numerical integration gives the value $-2{\cdot}08913$ for B_3. The boundary-layer velocity profile to this order is illustrated in figure 2.

4. The flow in the outer region

(*a*) *The perturbation field*

In the 'outer' region, i.e. away from the 'inner' boundary-layer region near $z = 0$, ψ is represented by an outer expansion

$$\psi = \psi_0(x,z) + \epsilon\psi_1(x,z) + \ldots \tag{4.1}$$

analogous to (3.4) Here ψ_0 is just the basic flow (2.3); ψ_1 and higher terms denote the perturbation of this basic flow due to its interaction with the boundary layer. Because the governing equations are elliptic this perturbation will extend also to the region $x < 0$ in the channel.

For $n = 1, 2$ and 3 the equations for $\psi_n(x,z)$ are, from (2.4) and (2.5),

$$\left.\begin{aligned} \psi_{0z}\psi_{nxz} - \psi_{0zz}\psi_{nx} &= -\Pi_{nx}, \\ -\psi_{0z}\psi_{nxx} &= -\Pi_{nz}, \end{aligned}\right\} \tag{4.2}$$

where $\Pi_n(x,z)$ denotes the nth term in the outer expansion of

$$\Pi(x,z) = P(x,z) + 4\epsilon^3 x. \tag{4.3}$$

The last term here represents the effect of the viscous terms in (2.4) and (2.5). Eliminating Π_n from (4.2), we obtain

$$\nabla^2\psi_{nx} - \frac{\psi_{0zzz}}{\psi_{0z}}\psi_{nx} = 0, \tag{4.4}$$

where $$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}.$$

We write $v = -\psi_{nx}$ and consider the following boundary-value problem in $-\infty < x < \infty$, $0 \leqslant z \leqslant \frac{1}{2}$:

$$\left.\begin{aligned} \nabla^2 v + \frac{2}{z(1-z)}v &= 0, \\ v(x,\tfrac{1}{2}) &= 0, \\ v(x,0) &= 0 \quad \text{for} \quad x < 0 \\ &= -\lambda \quad \text{for} \quad x > 0, \\ v \quad \text{bounded as} &\quad |x| \to \infty. \end{aligned}\right\} \tag{4.5}$$

These are the conditions satisfied by $-\psi_{nx}$; the matching of §5 gives $\lambda = 0$ for $n = 1, 2$ and $\lambda = 2$ for $n = 3$.

The construction of $v(x, z)$ given below suggests that the solution of (4.5) is unique. From this it follows that ψ_1 and ψ_2 vanish identically; we go on to consider ψ_3.

It will appear that, in the case $n = 3$, $v(x, z)$ is bounded as $x \to \infty$ and is exponentially small as $x \to -\infty$. Then the Fourier transform

$$V(\alpha, z) = \int_{-\infty}^{\infty} v(x, z)\, e^{i\alpha x} dx \tag{4.6}$$

converges in the open strip $0 < \operatorname{Im} \alpha < \mu$, where μ is some positive number. Introducing this into (4.5) we have

$$\left.\begin{aligned} \frac{d^2V}{dz^2} - \left\{\alpha^2 - \frac{2}{z(1-z)}\right\} V &= 0, \\ V(\alpha, 0) &= -2i\alpha^{-1}, \\ V(\alpha, \tfrac{1}{2}) &= 0. \end{aligned}\right\} \tag{4.7}$$

The equation for V is actually the special case $c = 0$ of the Rayleigh (inviscid Orr–Sommerfeld) equation

$$\frac{d^2w}{dz^2} - \left\{\alpha^2 + \frac{U''(z)}{U(z) - c}\right\} w = 0, \tag{4.8}$$

familiar in the context of hydrodynamic stability.

Two independent solutions of our equation can be found by the method of Frobenius; they can be characterized by their behaviour in the neighbourhood of $z = 0$:

$$\left.\begin{aligned} r(\alpha, z) &\sim z, \\ s(\alpha, z) &\sim 1 + 2z \ln z. \end{aligned}\right\} \tag{4.9}$$

The solution to (4.7) is then

$$V(\alpha, z) = \frac{2i}{\alpha}\left\{\frac{s(\alpha, \frac{1}{2})}{r(\alpha, \frac{1}{2})} r(\alpha, z) - s(\alpha, z)\right\}. \tag{4.10}$$

This has a pole at $\alpha = 0$ with residue $-2iV_0(z)$, where $V_0(z)$ is given by

$$\left.\begin{aligned} V_0''(z) + \frac{2}{z(1-z)} V_0(z) &= 0, \\ V_0(0) = 1,\ V_0(\tfrac{1}{2}) &= 0; \end{aligned}\right\} \tag{4.11}$$

the solution of this can be written down explicitly:†

$$V_0(z) = 1 - 2z - 2z(1-z) \ln \frac{z}{1-z}. \tag{4.12}$$

The remaining singularities of $V(\alpha, z)$ are the zeros of $r(\alpha, \frac{1}{2})$; these are the points $\pm i\beta_n$ $(\beta_n > 0)$, where β_n^2 are the eigenvalues of the problem

$$\left.\begin{aligned} w_n''(z) + \left\{\beta_n^2 + \frac{2}{z(1-z)}\right\} w_n(z) &= 0, \\ w_n(0) = w_n(\tfrac{1}{2}) &= 0; \end{aligned}\right\} \tag{4.13}$$

† In the case $\alpha = c = 0$ one solution of (4.8) is clearly $U(z)$; a second can be found by standard methods.

the residues are multiples of the $w_n(z)$. Comparison of (4.13) with (4.11), whose solution is monotonic in $[0, \frac{1}{2}]$, shows that all the eigenvalues β_n are real. The first five are 5·175, 11·938, 18·393, 24·769, 31·112. The corresponding eigenfunctions (normalized so that $w_n'(0) = 1$) are tabulated in table 2.

	$\beta_1 =$ 5·175, $A_1 = -12·94$	$\beta_2 =$ 11·938, $A_2 = -9·58$	$\beta_3 =$ 18·393, $A_3 = -8·92$	$\beta_4 =$ 24·769, $A_4 = -8·64$	$\beta_5 =$ 31·112, $A_5 = -8·49$
z	$w_1(z)$	$w_2(z)$	$w_3(z)$	$w_4(z)$	$w_5(z)$
0·02	0·01956	0·01941	0·01916	0·01880	0·01835
0·04	0·03811	0·03693	0·03497	0·03233	0·02910
0·06	0·05545	0·05157	0·04538	0·03742	0·02835
0·08	0·07140	0·06256	0·04908	0·03299	0·01658
0·10	0·08581	0·06931	0·04568	0·02028	−0·00158
0·12	0·09853	0·07152	0·03576	0·00251	−0·01914
0·14	0·10945	0·06913	0·02080	−0·01588	−0·02939
0·16	0·11847	0·06236	0·00291	−0·03036	−0·02844
0·18	0·12552	0·05168	−0·01539	−0·03736	−0·01667
0·20	0·13054	0·03779	−0·03155	−0·03518	0·00143
0·22	0·13351	0·02157	−0·04333	−0·02438	0·01899
0·24	0·13444	0·00402	−0·04912	−0·00760	0·02935
0·26	0·13333	−0·01377	−0·04812	0·01103	0·02859
0·28	0·13024	−0·03073	−0·04049	0·02697	0·01700
0·30	0·12524	−0·04582	−0·02728	0·03633	−0·00103
0·32	0·11842	−0·05814	−0·01032	0·03681	−0·01867
0·34	0·10990	−0·06695	0·00805	0·02832	−0·02924
0·36	0·09982	−0·07173	0·02532	0·01292	−0·02876
0·38	0·08831	−0·07219	0·03912	−0·00563	−0·01740
0·40	0·07557	−0·06830	0·04756	−0·02281	0·00053
0·42	0·06176	−0·06032	0·04947	−0·03443	0·01827
0·44	0·04709	−0·04872	0·04461	−0·03766	0·02910
0·46	0·03176	−0·03420	0·03363	−0·03172	0·02894
0·48	0·01599	−0·01763	0·01805	−0·01806	0·01784
0·50	0·00000	0·00000	0·00000	0·00000	0·00000

TABLE 2. The first five eigenfunctions $w_n(z)$

We can now obtain $v(x, z)$ from the inversion integral

$$v(x,z) = \frac{1}{2\pi}\int_{-\infty+i\mu_*}^{\infty+i\mu_*} V(\alpha,z)\,e^{-i\alpha x}\,d\alpha, \tag{4.14}$$

where $0 < \mu_* < \mu$; μ may be identified with β_1. For $x > 0$ the contour of integration is completed in the lower half-plane, and we obtain

$$v(x,z) = -2V_0(z) + \sum_{n=1}^{\infty} A_n e^{-\beta_n x} w_n(z), \tag{4.15}$$

where the A_n are constants; actually

$$A_n = \frac{2i}{\beta_n} s(-i\beta_n, \tfrac{1}{2}) \Big/ \frac{\partial r}{\partial \alpha}(-i\beta_n, \tfrac{1}{2}). \tag{4.16}$$

In the case $x < 0$, the result is

$$v(x,z) = -\sum_{n=1}^{\infty} A_n e^{\beta_n x} w_n(z). \tag{4.17}$$

The A_n are best found from the condition that v and $\partial v/\partial x$ be continuous at $x = 0$; this gives

$$V_0(z) = \sum_{n=1}^{\infty} A_n w_n(z), \tag{4.18}$$

so that the A_n are the Fourier coefficients in the expansion of $V_0(z)$,† whence

$$A_n = \left\{\int_0^{\frac{1}{2}} V_0(z)\, w_n(z)\, dz\right\} \Big/ \left\{\int_0^{\frac{1}{2}} w_n^2(z)\, dz\right\}. \tag{4.19}$$

The first five are $-12{\cdot}94$, $-9{\cdot}58$, $-8{\cdot}92$, $-8{\cdot}64$, $-8{\cdot}49$.

We can also obtain an expression for the pressure in the outer region. From (4.2),

$$\begin{aligned} P_{3z}(x,z) &= 2(z-z^2)\psi_{3xx} \\ &= 2(z-z^2) \sum_{n=1}^{\infty} A_n \beta_n e^{-\beta_n x} w_n(z) \\ &= -2 \sum_{n=1}^{\infty} \frac{A_n}{\beta_n} e^{-\beta_n x}\{(z-z^2) w_n''(z) + 2w_n(z)\}, \end{aligned} \tag{4.20}$$

for $x > 0$, where (4.13) has been used. Integrating the first term on the right of (4.20) by parts, we obtain

$$P_3(x,z) = -2 \sum_{n=1}^{\infty} \frac{A_n}{\beta_n} e^{-\beta_n x}\{(z-z^2)\, w_n'(z) - (1-2z)\, w_n(z)\}, \tag{4.21}$$

where the condition

$$P_3(x,0) = 0, \tag{4.22}$$

required for matching to the inner solution, has been used.

For $x < 0$ the corresponding expressions are

$$P_{3z}(x,z) = -2 \sum_{n=1}^{\infty} \frac{A_n}{\beta_n} e^{\beta_n x}\{(z-z^2)\, w_n''(z) + 2w_n(z)\}, \tag{4.23}$$

$$P_3(x,0) = -4x, \tag{4.24}$$

$$P_3(x,z) = -4x - 2 \sum_{n=1}^{\infty} \frac{A_n}{\beta_n} e^{\beta_n x}\{(z-z^2)\, w_n'(z) - (1-2z)\, w_n(z)\}. \tag{4.25}$$

To obtain (4.24), consider the form taken by the first equation in (4.2) when $z = 0$ and $x < 0$; this reduces to

$$\Pi_{3x}(x,z) \equiv P_{3x}(x,z) + 4 = 0. \tag{4.26}$$

The resulting constant of integration vanishes since, from (4.22), $P_3(0,0) = 0$.

The pressure distribution along the line of symmetry $z = \frac{1}{2}$ is shown in figure 3.

† That the spectrum of (4.13) is discrete is ensured by an argument analogous to that given by Titchmarsh (1962, p. 125).

(b) *The contraction ratio of the jet*

We conclude this section by utilizing our outer solution to compute the final velocity W and the final width χ of the jet. The non-dimensional momentum equation may be written in integral form as

$$\int_C \{P_{ij} - u_i u_j\} n_j \, dl = 0, \tag{4.27}$$

where dl denotes a line element of the closed curve C, which we take to consist of the following arcs: (i) a line $x = x_1 < 0$, $0 \leqslant z \leqslant \frac{1}{2}$ upstream,† (ii) a segment

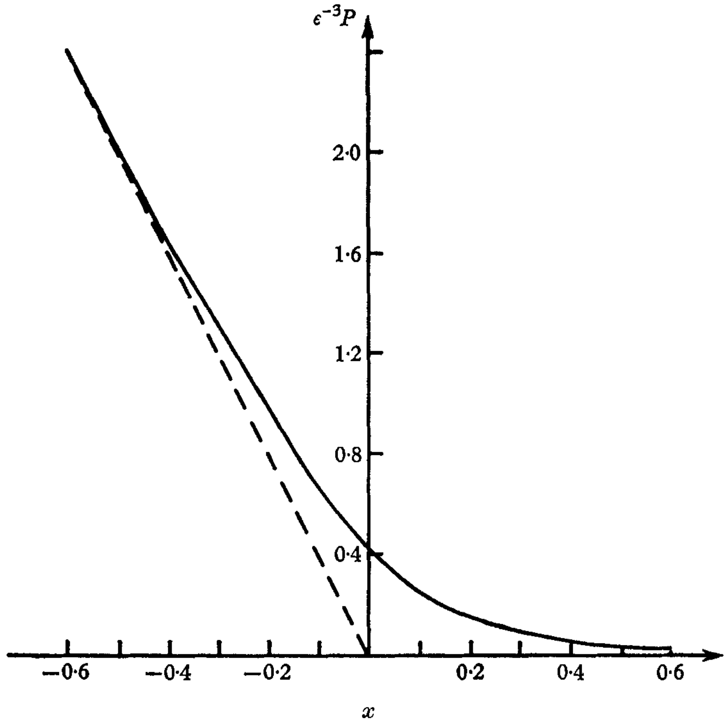

FIGURE 3. ———, the pressure distribution on the line of symmetry with $\epsilon = 0{\cdot}1$ (Reynolds number 1000); — — — —, case of infinite Reynolds number.

$x_1 \leqslant x \leqslant 0$, $z = 0$ of the lower wall, (iii) a segment $0 \leqslant x \leqslant x_2$, $z = \zeta(x)$ of the lower free streamline, (iv) a line $x = x_2$, $\zeta(x_2) \leqslant z \leqslant \frac{1}{2}$ downstream, (v) the segment $x_1 \leqslant x \leqslant x_2$, $z = \frac{1}{2}$ of the centre line. We take the x-component of (4.27) and note that (iii) and (v) make no contribution; in the limit $x_2 \to \infty$ we obtain

$$\int_0^{\frac{1}{2}} \{(P - 2\epsilon^3 \psi_{xz}) + \psi_z^2\}|_{x=x_1} dz - \int_{x_1}^0 \epsilon^3 \psi_{zz}(x, 0)\, dx - \tfrac{1}{2} W^2 \chi = 0. \tag{4.28}$$

But
$$P(x_1, z) = \epsilon^3\{-4x_1 + \Pi_3(x, z)\} + o(\epsilon^3), \tag{4.29}$$

and
$$\epsilon^3 \psi_{zz}(x, 0) = \epsilon^3 \psi_{0zz}(x, 0) + o(\epsilon^3) = 2\epsilon^3 + o(\epsilon^3); \tag{4.30}$$

† Since ψ_3 does not satisfy the no-slip condition on $z = 0$, $x < 0$, there will be a boundary layer near there where our outer solution is not valid. However, this will not contribute to (4.27) in the approximation to which we work.

there is thus a cancellation between the pressure $-4\epsilon^3 x_1$ of the basic Poiseuille flow and its integrated skin friction, leaving

$$\tfrac{1}{2}W^2\chi = \int^{\frac{1}{2}} \{\epsilon^3\Pi_3 - 2\epsilon^3\psi_{xz} + \psi_z^2\}|_{x=x_1} dz + o(\epsilon^3). \quad (4.31)$$

Now let $x_1 \to -\infty$: by (4.25) $\Pi_3 \to 0$, also $\psi \to \psi_0$. Hence†

$$\tfrac{1}{2}W^2\chi = \int_0^{\frac{1}{2}} 4(z-z^2)^2 dz + o(\epsilon^3)$$
$$= \tfrac{1}{15} + o(\epsilon^3). \quad (4.32)$$

We also have from conservation of mass

$$W\chi = \tfrac{1}{3}, \quad (4.33)$$

and hence we obtain the results

$$\left.\begin{aligned} \chi &= \tfrac{5}{6} + o(\epsilon^3), \\ W &= \tfrac{2}{5} + o(\epsilon^3). \end{aligned}\right\} \quad (4.34)$$

The $O(1)$ terms here are the two-dimensional analogues of the results obtaine by Harmon (1955), which are now seen to be correct in the limit of high Reynolds number; in fact they are correct to order ϵ^3.

5. The matching process and the determination of $h(x)$

We adopt the matching rule employed by Van Dyke (1964):

$$H_m E_n \psi = E_n H_m \psi, \quad (5.1)$$

where m, n are any integers. Here E_n denotes the $(n+1)$-term outer-expansion operator defined precisely by Fraenkel (1968); for our purposes we may take it to mean the operation: express in terms of the outer variables and truncate immediately after the term of order ϵ^n. H_m is the corresponding inner-expansion operator. It must be emphasized that for successful application of the matching rule (5.1) the stretching transformation between the inner and outer variables must be in the canonical form $y = \epsilon\eta$; thus the outer expansions must be written in terms of y, not z as in §4. If this precaution is not taken, (5.1) can be satisfied only approximately, and indeed in the higher-order matchings it cannot be satisfied at all; we require that the two expressions in (5.1) be exactly the same, for all m, n. We use the symbol $\hat{E}_n$ to denote outer expansions written in terms of z. Thus

$$\hat{E}_0\psi = \psi_0 = z^2 - \tfrac{2}{3}z^3$$
$$= (y+\epsilon h)^2 - \tfrac{2}{3}(y+\epsilon h)^3, \quad (5.2)$$

$$E_0\psi = y^2 - \tfrac{2}{3}y^3 = \epsilon^2\eta^2 - \tfrac{2}{3}\epsilon^3\eta^3, \quad (5.3)$$

$$H_2E_0\psi = \epsilon^2\eta^2 = y^2. \quad (5.4)$$

† The result (4.32) can also be obtained without letting $x_1 \to -\infty$ as follows:

$$\epsilon^3 \int_0^{\frac{1}{2}} \{\Pi_3 + 2\psi_{0z}\psi_{3z}\}|_{x=x_1} dz = 2\epsilon^3 \sum_{n=1}^{\infty} \frac{A_n}{\beta_n} e^{\beta_n x_1} \int_0^{\frac{1}{2}} \frac{d}{dz}\{(z-z^2)w_n(z)\}dz = 0;$$

also

$$-2\epsilon^3\psi_{xz} = o(\epsilon^3).$$

Thus (5.1) with $m = 2$, $n = 0$ gives the condition (3.12) for $\Psi_2(\xi, \eta)$. The asymptotic behaviour of Ψ_2,

$$\Psi_2 \sim \xi^{\frac{2}{3}}(\eta/\xi^{\frac{1}{3}} + c)^2 = (\epsilon^{-1}y + cx^{\frac{1}{3}})^2, \tag{5.5}$$

shows that

$$E_0 H_2 \psi = E_0(\epsilon^2 \Psi_2) = y^2 \tag{5.6}$$

as required.†

Similarly, taking $m = 3$, $n = 0$, we obtain the condition (3.22) for $\Psi_3(\xi, \eta)$, and the asymptotic expansion for Ψ_3 confirms that (5.1) is satisfied in this case also.

Next we consider (5.1) with $m = 2$, $n = 1$; logically this ought perhaps to have been done before finding $\Psi_3(\xi, \eta)$, but since $\psi_1(x, z)$ turns out to be identically zero this is unimportant. We have

$$\hat{E}_1 \psi = (y+\epsilon h)^2 - \tfrac{2}{3}(y+\epsilon h)^3 + \epsilon \psi_1(x, y+\epsilon h),$$

$$E_1 \psi = y^2 - \tfrac{2}{3}y^3 + \epsilon\{\psi_1(x,y) + 2(y-y^2)h_0(x)\}, \tag{5.7}$$

$$\begin{aligned} H_2 E_1 \psi &= \epsilon \psi_1(x,0) + \epsilon^2\{\eta^2 + \eta \psi_{1z}(x,0) + 2\eta h_0(x)\} \\ &= y^2 + \epsilon\{\psi_1(x,0) + y\psi_{1z}(x,0) + 2yh_0(x)\}, \end{aligned} \tag{5.8}$$

while by (5.5)

$$E_1 H_2 \psi = E_1(\epsilon^2 \Psi_2) = y^2 + 2\epsilon y c x^{\frac{1}{3}}. \tag{5.9}$$

In this case (5.1) gives us the boundary condition $\psi_1(x, 0) = 0$; the problem for $\psi_1(x, z)$ is now seen to be just the problem considered in §4 in the case $\lambda = 0$, and the solution is identically zero. The remaining terms in (5.8) and (5.9) then yield the result $h_0(x) = cx^{\frac{1}{3}}$ for the displacement of the free streamline, that is,

$$\zeta(x) = 0{\cdot}70798\epsilon x^{\frac{1}{3}} + O(\epsilon^2). \tag{5.10}$$

The first term in our outer expansion for ζ thus agrees in form with the result obtained by Goren (1966); the coefficient agrees to within 1 %.

The vanishing of $\psi_1(x, z)$ means that to order ϵ there is no interaction between the boundary layer and the outer flow: there is nothing analogous to the 'displacement effect' found in the case of flow along a rigid boundary. So the assumption made at the outset by Goren (1966) is justified. We can view this result as follows. Modifications to the outer flow are forced by the last term in (5.9): it is this that gives rise to the displacement effect in the solid-wall case. The difference between our case and that of a rigid boundary is that the term in question in (5.9) includes a streamwise velocity, while in the rigid-wall case the corresponding velocity is purely across the stream. It is to be expected physically that such a streamwise component of velocity cannot be dealt with by an essentially inviscid perturbation of the basic flow; instead the free surface adjusts itself so that the gain in flux due to speeding-up of the fluid in the boundary layer is cancelled by contraction of the jet.

The form of $h_0(x)$ obtained also ensures that (5.1) is satisfied for $m = 3$, $n = 1$. The next step is to determine $\psi_2(x, z)$ and $h_1(x)$ by considerations analogous to the above. We will not record the details here; the expressions for $E_3 H_3 \psi$ and $H_3 E_3 \psi$ given in appendix B serve as a check on the matching. The conclusion is that ψ_2 vanishes identically and $h_1(x) = \frac{1}{2}B_3 x^{\frac{2}{3}}$, that is,

$$\zeta(x) = 0{\cdot}70798\epsilon x^{\frac{1}{3}} - 1{\cdot}04457\epsilon^2 x^{\frac{2}{3}} + O(\epsilon^3). \tag{5.11}$$

† We may note in passing that $\hat{E}_0 H_2 \psi = z^2$, which is not identical with
$$H_2 \hat{E}_0 \psi = \epsilon^2(\eta + h_0)^2.$$

To obtain the boundary condition for $\psi_3(x, z)$, (5.1) is applied with $m = n = 3$. This yields $\psi_3(x, 0) = 2x$, so for the first time we have a non-trivial outer problem. To find $h_2(x)$ we must advance to $m = 4$; this is done in §7.

6. The composite expansion

Again following Van Dyke (1964), we introduce the 'composite expansion' defined by

$$C_n\psi = (E_n + H_n - E_n H_n)\psi. \tag{6.1}$$

This expression may be expected to provide a uniform approximation to order ϵ^n over the whole width of the jet; we shall apply it to a discussion of the overall momentum balance in the jet.

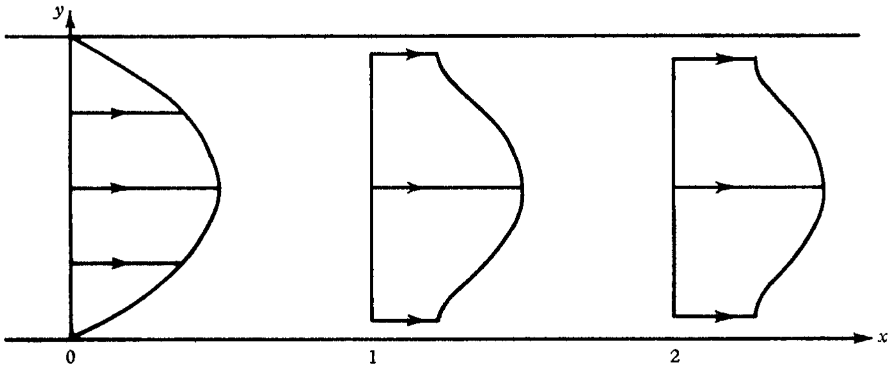

FIGURE 4. Velocity profiles based on the composite expansion. ($\epsilon = 0{\cdot}1$, Reynolds number 1000.)

From (B 1) and (B 3) we have

$$C_3\psi = \epsilon^2 x^{\frac{2}{3}} f_2(\theta) + \epsilon^3 x f_3(\theta) - 2\epsilon^2 y^2 h_1 + \epsilon^3\{2(y - y^2) h_2 - 4y h_0 h_1 + \psi_3(x, y) - 2x\}. \tag{6.2}$$

However, for our purposes it is more convenient to write this in terms of z rather than y:

$$C_3\psi = \epsilon^2 x^{\frac{2}{3}} f_2(\theta) + \epsilon^3 x f_3(\theta) - 2\epsilon^2 z^2 h_1 + \epsilon^3\{2(z - z^2) h_2 + \psi_3(x, z) - 2x\} + O(\epsilon^4),$$

where $O(\epsilon^4)$ denotes terms *uniformly* of the order indicated, because the meaning of θ has not been changed, and the O-term comes only from the third and subsequent terms of (6.2). A similar expression may be written down for the velocity $u = \psi_z$; working to second order only this is

$$C_2 u = \epsilon x^{\frac{1}{3}} f_2'(\theta) + \epsilon^2 x^{\frac{2}{3}} f_3'(\theta) - 2\epsilon^2 B_3 z x^{\frac{2}{3}} + O(\epsilon^3). \tag{6.3}$$

Velocity profiles based on this expression are shown in figure 4.

As a valuable check on our procedures we shall calculate the momentum integral (4.27) along the path (iv) (without letting $x_2 \to \infty$). This is

$$M = \int_{\zeta}^{\frac{1}{2}} \{(P - 2\epsilon^3 \psi_{xz}) + \psi_z^2\}|_{x=x_2} dz, \tag{6.4}$$

and we may expand it as

$$M = M_0 + \epsilon^3 M_3 + o(\epsilon^3). \tag{6.5}$$

We have

$$C_3 P = P_3(x, z) + O(\epsilon^4), \tag{6.6}$$

$$\begin{aligned} C_3 \psi_z^2 = {} & 4(z-z^2)^2 + 4\epsilon^3(z-z^2)\,\psi_{3z} \\ & + \epsilon^2 x^{\frac{2}{3}}\{f_2'^2(\theta) - 4(\theta+c)^2\} \\ & + 2\epsilon^3 x\{f_2'(\theta) f_3'(\theta) + 4(\theta+c)^3 - 2B_3(\theta+c)\} \\ & + O(\epsilon^4). \end{aligned} \tag{6.7}$$

These expressions give

$$M_0 = \int_0^{\frac{1}{2}} 4(z-z^2)^2 dz = \tfrac{1}{15}, \tag{6.8}$$

and

$$\begin{aligned} M_3 = {} & -\int_0^{cx^{\frac{1}{3}}} 4z^2 dz + \int_0^{\frac{1}{2}} \{P_3 + 4(z-z^2)\psi_{3z}\} dz \\ & + x\int_0^{\infty} \{f_2'^2(\theta) - 4(\theta+c)^2\} d\theta. \end{aligned} \tag{6.9}$$

The first term here is to account for the fact that the integration starts at $z = \zeta$, not $z = 0$. The first and last terms in (6.9) may be integrated at once to give $-\frac{4}{3}c^3 x$ and $(2 + \frac{4}{3}c^3)x$, respectively; to calculate the second term we note that

$$P_3 + 4(z-z^2)\psi_{3z} = 8xV_0(z) + 2\sum_{n=1}^{\infty} \frac{A_n}{\beta_n} e^{-\beta_n x} \frac{d}{dz}\{(z-z^2)\, w_n(z)\}, \tag{6.10}$$

so that

$$\begin{aligned} \int_0^{\frac{1}{2}} \{P_3 + 4(z-z^2)\psi_{3z}\} dz &= 8x\int_0^{\frac{1}{2}} (z-z^2) V_0'(z)\, dz \\ &= -2x. \end{aligned} \tag{6.11}$$

Therefore M_3 vanishes, and momentum balance is satisfied.

7. The higher-order boundary layer

The expression (4.15) gives us, for $x > 0$,

$$\psi_3(x, z) = 2xV_0(z) + \sum_{n=1}^{\infty} \frac{A_n}{\beta_n} e^{-\beta_n x} w_n(z), \tag{7.1}$$

and hence

$$\begin{aligned} H_4 E_3 \psi = {} & H_3 E_3 \psi - 2\epsilon^2 y^2 h_1 \\ & + \epsilon^3 \left\{ -4yh_0 h_1 - 4xy + 2yh_2 - 4xy \ln y + y \sum_{n=1}^{\infty} \frac{A_n}{\beta_n} e^{-\beta_n x} \right\}. \end{aligned} \tag{7.2}$$

The normalizing condition $w_n'(0) = 1$ has been used here. This gives us the boundary condition on $\Psi_4(\xi, \eta)$ for $\eta \to \infty$. It should be noted, however, that the expression (7.2) when written in terms of the inner variable η contains a term in $\epsilon^4 \ln \epsilon$ as well as one in ϵ^4. Therefore we must expect that from now on our expansions for ψ and h will include terms in $\epsilon^n \ln \epsilon$ as well as ϵ^n. Experience with singular-perturbation problems has shown that in these circumstances it is not generally possible to achieve a match if the term in $\epsilon^n \ln \epsilon$ is treated as a separate term in the expansion; instead the ϵ^n and $\epsilon^n \ln \epsilon$ terms must be taken together. The theoretical basis for this is discussed by Fraenkel (1968).

It turns out that a logarithmic term is needed in the expansion for h, but not, at this stage at any rate, in the inner expansion for ψ. Proceeding then with the latter expansion, the equation for $\Psi_4(\xi,\eta)$ is, from (3.2),

$$\Psi_{4\eta\eta\eta}+\Psi_{2\xi}\Psi_{4\eta\eta}+\Psi_{2\eta\eta}\Psi_{4\xi}-\Psi_{2\eta}\Psi_{4\xi\eta}-\Psi_{2\xi\eta}\Psi_{4\eta} = F+G, \tag{7.3}$$

where

$$F = \Psi_{3\eta}\Psi_{3\xi\eta}-\Psi_{3\xi}\Psi_{3\eta\eta},$$
$$G = P_{4\xi}-h_0'P_{4\eta}-\Psi_{2\xi\xi\eta}+h_0''\Psi_{2\eta\eta}+2h_0'\Psi_{2\xi\eta\eta}-h_0'^2\Psi_{2\eta\eta\eta}.$$

We must now evaluate F and G in terms of f_2 and f_3; this is done in appendix C. The expressions obtained there lead us to write $\Psi_4(\xi,\eta)$ in the form

$$\Psi_4(\xi,\eta) = x^{\frac{4}{3}}f_{4a}(\theta)+x^{-\frac{2}{3}}f_{4b}(\theta). \tag{7.4}$$

$f_{4a}(\theta)$ and $f_{4b}(\theta)$ then satisfy

$$f_{4a}'''+\tfrac{2}{3}f_2f_{4a}''-\tfrac{4}{3}f_2'f_{4a}'+\tfrac{4}{3}f_2''f_{4a} = \tfrac{2}{3}f_3'^2-f_3f_3'', \tag{7.5}$$

$$f_{4b}'''+\tfrac{2}{3}f_2f_{4b}''+\tfrac{2}{3}f_2'f_{4b}'-\tfrac{2}{3}f_2''f_{4b} = g. \tag{7.6}$$

The boundary conditions at $\eta = 0$ are, from (3.7) and (3.9), $\Psi_4 = 0$ and

$$\Psi_{4\eta\eta} = 2h_0'\Psi_{2\xi\eta}-h_0''\Psi_{2\eta}. \tag{7.7}$$

In terms of f_{4a} and f_{4b} these become

$$f_{4a}(0) = f_{4a}''(0) = 0,$$
$$f_{4b}(0) = 0, \quad f_{4b}''(0) = \tfrac{4}{9}cf_2'(0). \tag{7.8}$$

The asymptotic behaviour of the right-hand side of (7.5) for large θ, or large $t = \theta+c$, is given by

$$\tfrac{2}{3}f_3'^2-f_3f_3'' \sim \tfrac{4}{3}B_3t^2+8t+\tfrac{2}{3}B_3^2+O(e^{-\frac{2}{9}t^3}). \tag{7.9}$$

Direct substitution in (7.5) gives us a particular integral whose asymptotic form is

$$-B_3t^2-4t\ln t+\tfrac{1}{4}B_3^2+O(t^{-2}).$$

To find the asymptotic behaviour of $f_{4a}(\theta)$, we must add to this expression the asymptotic solution of the homogeneous equation corresponding to (7.5); this is

$$A_{4a}(t^4+12t\ln t)+B_{4a}t+O(e^{-\frac{2}{9}t^3}),$$

where A_{4a}, B_{4a} are arbitrary constants. We take $A_{4a} = 0$ and choose B_{4a} and the third arbitrary constant in the exponential term so as to satisfy the boundary conditions (7.8). The choice of A_{4a} is dictated by the form of (7.2). We thus arrive at

$$f_{4a} \sim -B_3t^2-4t\ln t+B_{4a}t+\tfrac{1}{4}B_3^2+O(t^{-2}). \tag{7.10}$$

The corresponding results for (7.6) are

$$g \sim \tfrac{8}{27}f_2'(0)+O(e^{-\frac{2}{9}t^3}), \tag{7.11}$$

giving asymptotically a particular integral

$$-\tfrac{2}{9}f_2'(0)+O(e^{-\frac{2}{9}t^3})$$

and complementary function

$$A_{4b}\{t^{-2}+O(t^{-5})\}+B_{4b}t+O(e^{-\frac{2}{9}t^3}).$$

The two conditions (7.8) must be satisfied, but it is not clear what the third boundary condition should be. Probably A_{4b} will be related by a higher-order matching to the arbitrary constants multiplying the eigensolutions discussed in §8; the indeterminacy reflects our ignorance of conditions in the neighbourhood of $x = 0$. We have, in any case

$$f_{4b} \sim B_{4b}t-\tfrac{2}{9}f_2'(0)+O(t^{-2}). \tag{7.12}$$

The asymptotic behaviour we have found for $\Psi_4(\xi,\eta)$ gives

$$\begin{aligned} E_3H_4\psi = E_3H_3\psi &- \epsilon^2 B_3 x^{\frac{2}{3}} y^2 - 2\epsilon^3 B_3 cxy \\ &- 4\epsilon^3 xy(\ln y - \ln\epsilon - \tfrac{1}{3}\ln x) \\ &+ \epsilon^3 y(B_{4a}x + B_{4b}x^{-1}). \end{aligned} \tag{7.13}$$

Comparison with (7.2) shows that the matching is accomplished provided that

$$\begin{aligned} h_2(x) = (2+\tfrac{1}{2}B_{4a})\,x+\tfrac{2}{3}x\ln x+\tfrac{1}{2}B_{4b}x^{-1} \\ -\frac{1}{2}\sum_{n=1}^{\infty}\frac{A_n}{\beta_n}e^{-\beta_n x}+2x\ln\epsilon. \end{aligned} \tag{7.14}$$

As mentioned earlier, h now contains a term logarithmic in ϵ. The presence of a term in x^{-1} indicates that, as expected, our solution breaks down for small x as well as for large x.

It has not been thought worth while to discuss here the fifth-order terms, although no difficulties of principle arise.

8. Eigensolutions

An eigensolution of the inner problem is here defined to be a function $\Phi_m(\xi,\eta) = \xi^{-\lambda_m}\, g(\theta)$ satisfying

$$\left.\begin{aligned} g'''+\tfrac{2}{3}f_2 g''+\lambda_m f_2' g'-\lambda_m f_2'' g = 0, \\ g(0) = g''(0) = 0, \\ g = At+O(e^{-\frac{2}{9}t^3}) \quad \text{as} \quad t = \theta + c \to \infty, \end{aligned}\right\} \tag{8.1}$$

where A is a constant. Any multiple of such a function $\Phi_m(\xi,\eta)$ may be introduced at any stage into the inner expansion without disturbing the equation of motion or the boundary conditions on $\eta = 0$. The asymptotic behaviour $g \sim At$ is permitted since the corresponding matching can always be accomplished by a suitable adjustment to the free surface $h(x)$.

The indeterminacy that results from the possibility of eigensolutions entering is due physically to the existence of a region between the point where the jet leaves the wall of the channel and the point where our boundary-layer solution becomes valid, where the flow can only be described by the full Navier–Stokes equations, and the nature of the solution is unknown. A similar indeterminacy occurs in the higher-order theory of the boundary layer in the flow past a flat plate (Goldstein 1960); there the details of the flow near the leading edge of the plate are unknown.

Stewartson (1957) showed that in the flat-plate case there is an infinite number of such eigensolutions, and in our case similar arguments lead to the same conclusion. The simplest eigensolution, which is the least singular as $x \to 0$ with η fixed, is

$$\Phi_1(\xi, \eta) = x^{-\frac{1}{3}}\{f_2(\theta) - \tfrac{1}{2}\theta f_2'(\theta)\}. \tag{8.2}$$

This eigensolution corresponds to an uncertainty in the starting point of the boundary layer. Since, to lowest order, $\zeta'(x) = \frac{1}{3}c\epsilon x^{-\frac{2}{3}}$, an uncertainty Δx in the starting point gives rise to an uncertainty $\Delta\zeta \sim \frac{1}{3}c\epsilon x^{-\frac{2}{3}}\Delta x$ in ζ; the contribution to ζ of a term $\epsilon^{\kappa}\Phi_1(\xi, \eta)$ in ψ is $c\epsilon^{\kappa-1}x^{-\frac{2}{3}}$.

Now the boundary-layer thickness δ at the point x is of order $\epsilon x^{\frac{1}{3}}$; boundary-layer theory neglects viscous forces arising from velocity gradients in the x-direction, and so must be expected to break down when x and δ are of the same order, i.e. when $x = O(\epsilon^{\frac{3}{2}})$. Note that in this region $\zeta'(x) = O(1)$, as might have been expected.

We anticipate, therefore, an uncertainty $\Delta x = O(\epsilon^{\frac{3}{2}})$, giving $\Delta\zeta = O(\epsilon^{\frac{5}{2}}x^{-\frac{2}{3}})$. So we expect the eigensolution $\Phi_1(\xi, \eta)$ not to enter the inner expansion before order $\kappa = \frac{7}{2}$. This is consistent with the fact that our solution contains an indeterminacy in Ψ_4, but not earlier.

Appendix A

Here we give some details of the asymptotic solutions of the various ordinary differential equations that have arisen. In the equation for $f_2(\theta)$,

$$f_2''' + \tfrac{2}{3}f_2 f_2'' - \tfrac{1}{3}f_2'^2 = 0, \tag{A 1}$$

we set $t = \theta + c$, where c is as yet arbitrary, and

$$f_2(t) = \alpha t^2 + g(t), \tag{A 2}$$

where α is a constant. The function αt^2 satisfies (A 1), but not the boundary conditions associated with it; $g(t)$ represents the error arising from these, which we expect to be small for large t. (A 1) becomes

$$g''' + \tfrac{2}{3}\alpha t^2 g'' - \tfrac{4}{3}\alpha t g' + \tfrac{4}{3}\alpha g = 0, \tag{A 3}$$

where terms quadratic in g have been omitted. A plausible procedure is now to find a solution to (A 3) which is $o(t^2)$, and such that the neglected terms are $o(1)$; such a solution will then give an approximate solution to the full equation for g, and hence an asymptotic solution for f_2.

Two solutions of (A 3) are $g = t$ and $g = t^2$; to find a third, let $g(t) = th(t)$. Then

$$h''' + (3t^{-1} + \tfrac{2}{3}\alpha t^2)h'' = 0, \tag{A 4}$$

which has solution

$$h'' = Ct^{-3}e^{-\frac{2}{9}\alpha t^3}, \tag{A 5}$$

so that the general solution of (A 3) is

$$g(t) = At^2 + Bt + C\,e^{-\frac{2}{9}\alpha t^3}\left\{\frac{81}{4\alpha^2 t^6} + o(t^{-6})\right\}. \tag{A 6}$$

In this we take $A = B = 0$, giving an asymptotic solution of (A 1)

$$f_2(t) \sim \alpha t^2 + C\, e^{-\frac{2}{9}\alpha t^3}\left\{\frac{81}{4\alpha^2 t^6} + \ldots\right\}, \tag{A 7}$$

which contains three arbitrary constants α, c and C which we choose in the manner described in § 3.

The equation for $f_3(\theta)$,

$$f_3''' + \tfrac{2}{3}f_2 f_3'' - f_2' f_3' + f_2'' f_3 = 0, \tag{A 8}$$

is satisfied by f_2', so we write

$$f_3(t) = f_2'(t)\, p_3(t), \tag{A 9}$$

and then $p_3(t)$ satisfies

$$p_3''' + \left(\frac{3f_2''}{f_2'} + \tfrac{2}{3}f_2\right)p_3'' + \left(\frac{3f_2'''}{f_2'} + \frac{4}{3}\frac{f_2 f_2''}{f_2'} - f_2'\right)p_3' = 0. \tag{A 10}$$

This equation is linear, and so can be discussed rigorously, e.g. by the methods described by Jeffreys (1962): the details will not be given here since the method is standard. The same method may be used for $f_{4a}(\theta)$ and $f_{4b}(\theta)$.

Appendix B

We record here the details of the matching with $m = n = 3$, used at the end of § 5. We have

$$\begin{aligned} E_3\psi = {} & y^2 - \tfrac{2}{3}y^3 + 2\epsilon(y-y^2)h_0 \\ & + \epsilon^2\{2(y-y^2)h_1 + (1-2y)h_0^2\} \\ & + \epsilon^3\{2(y-y^2)h_2 + 2(1-2y)h_0h_1 - \tfrac{2}{3}h_0^3 + \psi_3(x,y)\}, \end{aligned} \tag{B 1}$$

$$\begin{aligned} H_3E_3\psi = {} & y^2 - \tfrac{2}{3}y^3 + 2\epsilon(y-y^2)h_0 + \epsilon^2\{2yh_1 + (1-2y)h_0^2\} \\ & + \epsilon^3\{-\tfrac{2}{3}h_0^3 + 2h_0h_1 + \psi_3(x,0)\}, \end{aligned} \tag{B 2}$$

$$\begin{aligned} E_3H_3\psi = {} & y^2 - \tfrac{2}{3}y^3 + 2\epsilon(y-y^2)cx^{\frac{1}{3}} + \epsilon^2\{(1-2y)c^2x^{\frac{2}{3}} + B_3x^{\frac{2}{3}}y\} \\ & + \epsilon^3\{-\tfrac{2}{3}c^3x + B_3cx + 2x\}. \end{aligned} \tag{B 3}$$

In deriving (B 3) the asymptotic expansion for $\eta \to \infty$,

$$\Psi_3 \sim -\tfrac{2}{3}(\epsilon^{-1}y + cx^{\frac{1}{3}})^3 + B_3x^{\frac{2}{3}}(\epsilon^{-1}y + cx^{\frac{1}{3}}) + 2x, \tag{B 4}$$

has been used. Comparison of (B 2) and (B 3) yields $\psi_3(x,0) = 2x$.

Appendix C

Here we evaluate the F and G of (7.3) in terms of f_2 and f_3. Substituting for Ψ_3 from (3.23) gives

$$F = \xi^{\frac{1}{3}}(\tfrac{2}{3}f_3'^2 - f_3 f_3''). \tag{C 1}$$

To find the corresponding expression for G we must first calculate the pressure; this is found from (3.3), together with the boundary condition (3.8). These give

$$\begin{aligned} P_{4\eta}(\xi,\eta) &= -\tfrac{1}{9}x^{-1}\{4f_2f_2' - t(f_2'^2 + 2f_2f_2'' + 3f_2''')\} \\ &= -\tfrac{2}{27}x^{-1}\frac{d}{d\theta}(4f_2^2 - 2tf_2f_2' - 3tf_2'' + 3f_2'), \end{aligned} \tag{C 2}$$

where $t = \theta + c$, and

$$P_4(\xi, 0) = -\tfrac{2}{3}x^{-\frac{2}{3}}f_2'(0). \tag{C 3}$$

Hence $$P_4(\xi, \eta) = -\tfrac{2}{27}x^{-\frac{2}{3}}(4f_2^2 - 2tf_2f_2' - 3tf_2'' - 3f_2') - \tfrac{4}{9}x^{-\frac{2}{3}}f_2'(0). \tag{C 4}$$

This gives $G = x^{-\frac{5}{3}}g(\theta)$, where

$$\begin{aligned} g = {} & \tfrac{1}{27}t(4f_2f_2' - 2tf_2'^2) \\ & + \tfrac{4}{81}(4f_2^2 - 2tf_2f_2' - 3tf_2'' + 3f_2') \\ & - \tfrac{2}{9}(tf_2'' - f_2' + \tfrac{1}{2}t^2f_2''') + \tfrac{8}{27}f_2'(0). \end{aligned} \tag{C 5}$$

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